PRINCIPLES OF ANALYSIS
LECTURE 12 - CAUCHY SEQUENCES

PAUL L. BAILEY

1. Cauchy Sequences

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that $\{s_n\}_{n=1}^{\infty}$ is a *Cauchy sequence* if

$$\forall \epsilon > 0 \; \exists N \in \mathbb{Z}^+ \; \ni \; m, n \geq N \Rightarrow |s_m - s_n| < \epsilon.$$

Proposition 1. *Let $\{s_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Then $\{s_n\}_{n=1}^{\infty}$ is bounded.*

Proof. Since $\{s_n\}_{n=1}^{\infty}$ is Cauchy, there exists $N \in \mathbb{Z}^+$ such that if $m, n \geq N$, then $|s_m - s_n| < 1$. In particular, for every $n \geq N$, we have $|s_n - s_N| < 1$. Set

$$M = \max\{s_1, s_2, \dots, s_{N-1}, s_N + 1\}.$$

Then $s_n \in [-M, M]$ for every $n \in \mathbb{Z}^+$. □

Proposition 2. *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then $\{s_n\}_{n=1}^{\infty}$ is convergent if and only if it is a Cauchy sequence.*

Proof. We prove each direction of the double implication.

($\Rightarrow$) Assume that the sequence is convergent. Let $\epsilon > 0$, and set $s = \lim s_n$. Then there exists $N \in \mathbb{Z}^+$ such that if $n \geq N$, then $|s_n - s| < \epsilon/2$. Then for $m, n \geq N$, we have

$$\begin{aligned} |s_m - s_n| &= |s_m - s + s - s_n| \\ &= |s_m - s| + |s_n - s| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

($\Leftarrow$) Assume that the sequence is a Cauchy sequence. Then it is bounded, and so its limit superior and inferior exist as real numbers. By a previous proposition, it suffices to show that $\liminf s_n = limsups_n$.

Let $\epsilon > 0$. Then there exists $N \in \mathbb{Z}^+$ such that if $m, n \geq N$, then $|s_m - s_n| < \epsilon$. In particular, $|s_n - s_N| < \frac{\epsilon}{2}$ for all $n \geq N$, so $s_N + \frac{\epsilon}{2}$ is an upper bound for $\{s_n \mid n \geq N\}$. Thus $\sup\{s_n \mid n \geq N\} \leq s_N + \frac{\epsilon}{2}$, and therefore $\limsup s_n \leq s_N + \frac{\epsilon}{2}$. Similarly $\liminf s_n \geq s_N - \frac{\epsilon}{2}$. Rearranging these inequalities gives

$$\limsup s_n - \frac{\epsilon}{2} \leq s_N \leq \liminf s_n + \frac{\epsilon}{2},$$

or

$$\limsup s_n - \liminf s_m < \epsilon.$$

Since ϵ is arbitrary, we have $\limsup s_n = \liminf s_n$. □

Date: September 29, 2003.

2. Subsequences

Let $s : \mathbb{Z}^+ \to \mathbb{R}$ be a sequence of real numbers. A *subsequence* of s is the composition $s \circ n$ of s with a strictly increasing sequence $n : \mathbb{Z}^+ \to \mathbb{Z}^+$ of positive integers.

If we denote the sequence s by $\{s_n\}_{n=1}^{\infty}$ and the sequence n by $\{n_k\}_{k=1}^{\infty}$, then we denote the subsequence by $\{s_{n_k}\}_{k=1}^{\infty}$.

Note that since the function $n : \mathbb{Z}^+ \to \mathbb{Z}^+$ is strictly increasing, it is injective. Thus if $N \in \mathbb{Z}^+$, there exists $k \in \mathbb{Z}^+$ such that $n_k \geq N$; otherwise, we would have an injective function from an infinite set into the finite set $\{m \in \mathbb{Z}^+ \mid m < N\}$.

Proposition 3. *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers and let $s \in \mathbb{R}$. Then $\{s_n\}_{n=1}^{\infty}$ converges to s if and only if every subsequence of $\{s_n\}_{n=1}^{\infty}$ converges to s.*

Proof. We prove both directions.

($\Leftarrow$) Note that a sequence is a subsequence of itself. Thus if every subsequence of $\{s_n\}_{n=1}^{\infty}$ converges to s, then in particular the sequence itself converges to s.

($\Rightarrow$) Suppose that $\lim s_n = s$. Let $\{s_{n_k}\}$ be a subsequence of $\{s_n\}_{n=1}^{\infty}$, and let $\epsilon > 0$. Then there exists $N \in \mathbb{Z}^+$ such that if $n \geq N$, then $|s_n - s| < \epsilon$. We know that there exists $K \in \mathbb{Z}^+$ such that $n_K \geq N$; moreover, since $\{n_k\}$ is strictly increasing, if $k \geq K$, then $n_k \geq n_K \geq N$. Therefore, for $k \geq K$, we have $|s_{n_k} - s| < \epsilon$. □

Proposition 4. *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then $\{s_n\}_{n=1}^{\infty}$ has a monotonic subsequence.*

Proof. Say that the i^{th} term of $\{s_n\}_{n=1}^{\infty}$ is *dominant* if $s_j < s_i$ for every $j > i$.

Case 1: There are infinitely many dominant terms. In this case, set

$$n_1 = \min\{n \in \mathbb{Z}^+ \mid s_n \text{ is dominant}\}.$$

Then recursively set

$$n_{k+1} = \min\{n \in \mathbb{Z}^+ \mid s_n \text{ is dominant and } n > n_k\};$$

this set is nonempty by the hypothesis of this case. Then $\{s_{n_k}\}$ is a decreasing sequence.

Case 2: There are finitely many dominant terms. In this case, set

$$n_0 = max\{n \in \mathbb{Z}^+ \mid s_n \text{ is dominant}\}.$$

Then recursively set

$$n_{k+1} = \min\{n \in \mathbb{Z}^+ \mid s_n > s_{n_k} \text{ and } n > n_k\};$$

this set is nonempty because s_{n_0} was the last dominant term. Now $\{s_{n_k}\}$ is an increasing sequence. □

Corollary 1. *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. It is clear that if a sequence is bounded, then every subsequence is also bounded. Thus a bounded sequence has a bounded monotonic subsequence, which must converge. □

3. Cluster Points and Subsequential Limits

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and let $c \in \mathbb{R}$.

We say that c is a *cluster point* of $\{s_n\}_{n=1}^{\infty}$ if

$$\forall \epsilon > 0 \; \forall N \in \mathbb{Z}^+ \; \exists n \geq N \;\ni\; |s_n - c| < \epsilon.$$

We say that c is a *subsequential limit* of $\{s_n\}_{n=1}^{\infty}$ if there exists a subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} s_{n_k} = c$.

Proposition 5. *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and let $c \in \mathbb{R}$. Then c is a cluster point if and only if c is a subsequential limit.*

Proof. Exercise. □

4. Neighborhoods

Let $x_0 \in \mathbb{R}$. An *ϵ-neighborhood* of x_0 is an open interval of the form $(x_0 - \epsilon, x_0 + \epsilon)$, where $\epsilon > 0$.

More generally, a *neighborhood* of x_0 is a subset $Q \subset \mathbb{R}$ such that there exists $\epsilon > 0$ with $(x_0 - \epsilon, x_0 + \epsilon) \subset Q$.

A *deleted neighborhood* of x_0 is a set of the form $Q \setminus \{x_0\}$, where Q is a neighborhood of x_0.

Proposition 6. *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and let $s \in \mathbb{R}$. Then s is the limit of $\{s_n\}_{n=1}^{\infty}$ if and only if every neighborhood of s contains s_n for all but finitely many n.*

Proof. Gaughan page 35 Lemma. □

Proposition 7. *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and let $c \in \mathbb{R}$. Then c is a cluster point of $\{s_n\}_{n=1}^{\infty}$ if and only if every neighborhood of c contains s_n for infinitely many n.*

Proof. Exercise. □

Department of Mathematics and CSCI, Southern Arkansas University
E-mail address: plbailey@saumag.edu